Higher order Toda brackets

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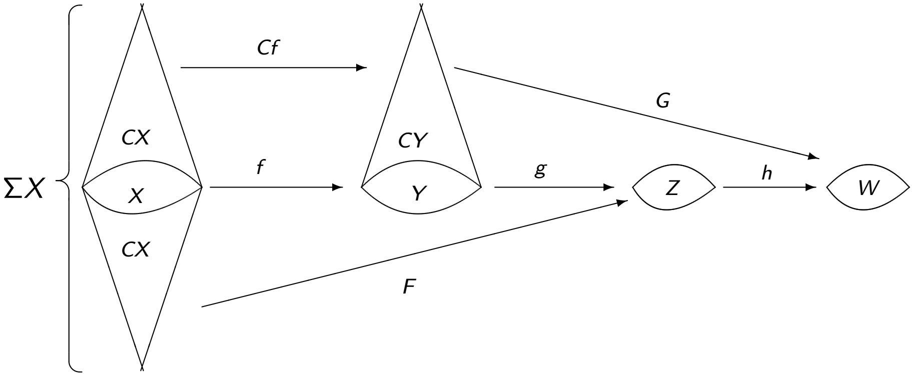

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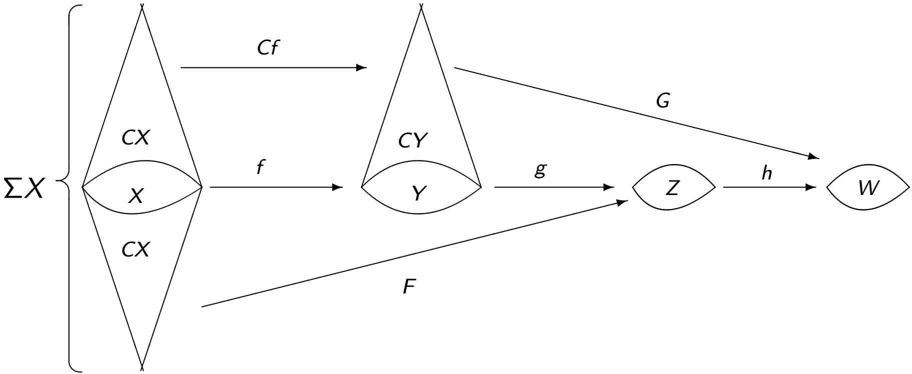

The nullhomotopies $h \circ F$ and $G \circ Cf$ induce a map

$$\langle f, g, h, (F, G) \rangle : \Sigma X \to W$$

Toda bracket and rectification of linear diagrams

Definition: Given a linear diagram

$X_* = (X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \dots X_{n-1} \xrightarrow{f_{n-1}} X_n)$ where $f_{j+1} \circ f_j \sim *$,

X_* called *rectifiable* if we have:

$$\begin{array}{ccccccccc}
X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & \dots & \longrightarrow & X_{n-1} & \xrightarrow{f_{n-1}} & X_n \\
\downarrow\simeq & & \downarrow\simeq & & & & \downarrow\simeq & & \downarrow\simeq \\
X'_1 & \xrightarrow{f'_1} & X'_2 & \xrightarrow{f'_2} & \dots & \longrightarrow & X'_{n-1} & \xrightarrow{f'_{n-1}} & X'_n \\
\uparrow\simeq & & \uparrow\simeq & & & & \uparrow\simeq & & \uparrow\simeq \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
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Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & \dots & \longrightarrow & Y_{n-1} & \xrightarrow{g_{n-1}} & Y_n
\end{array}$$

where $g_{j+1} \circ g_j = *$.

Example in the category of chain complexes

Let $X_* \xrightarrow{f_*} Y_* \xrightarrow{g_*} Z_* \xrightarrow{h_*} W_*$ be a linear diagram in Ch:

$$
\begin{array}{ccccccc}
\cdots & & \cdots & & \cdots & & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{4.} & \mathbb{Z} \\
\downarrow & & \downarrow & & \downarrow {\scriptstyle 8.} & & \downarrow {\scriptstyle 2.} \\
\mathbb{Z} & \xrightarrow{1.} & \mathbb{Z} & \xrightarrow{2.} & \mathbb{Z} & \xrightarrow{1.} & \mathbb{Z} \\
\downarrow {\scriptstyle 2.} & & \downarrow {\scriptstyle 4.} & & \downarrow & & \downarrow \\
\mathbb{Z} & \xrightarrow{2.} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
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\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & & \cdots & & \cdots & & \cdots
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Toda bracket and rectification of linear diagrams(cont.)

Theorem: For $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$, where $g \circ f \sim *$ and $h \circ g \sim *$. There are null-homotopies $F : CX \to Z$, and $G : CY \to W$, such that $\langle f, g, h, (F, G) \rangle$ is nullhomotopic if and only if the diagram $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ is rectifiable.

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$$\{ \, [\langle f, g, h, (F, G)\rangle] \mid F : g \circ f \simeq *, G : h \circ g \simeq *\}$$

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Fact: Given $X_* \xrightarrow{f_*} Y_* \xrightarrow{g_*} Z_* \xrightarrow{h_*} W_*$, then

$$\langle f_*, g_*, h_*\rangle = [\langle f_*, g_*, h_*, (F_*, G_*)\rangle] + h_*[\Sigma X_*, Z_*] + [\Sigma Y_*, W_*]\Sigma f_*$$

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In our example $h_*[\Sigma X_*, Z_*] = 0$, and $[\Sigma Y_*, W_*]\Sigma f_* = 0$.
In addition, one can find a value of Toda bracket equal to $[\mathrm{Id}]$.

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Homotopy cofiber of a map

Definition: The homotopy cofiber of a map $f : X \to Y$ is the pushout of

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The homotopy cofiber is "homotopy invariant".

Toda bracket in terms of homotopy cofiber

If we have $X \xrightarrow{f} Y \xrightarrow{g} Z$ and $F : CX \to Z$ a nullhomotopy for $g \circ f$, then we have the following commutative square:

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The composition equal to $\langle f, g, h, (F, G)\rangle : \Sigma X \to W$.

Higher order Toda bracket

To extend the definition of Toda bracket to longer sequence $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W \xrightarrow{k} V,$ we not just need the compositions to be nullhomotopic , we need also the lower order Toda brackets ($\langle f, g, h, (F, G)\rangle$ and $\langle g, h, k, (G, H)\rangle$) to be nullhomotopic.

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In this talk we will give new description using diagrams and homotopy cofiber of cubes.

Diagrammatic description for Toda system

The maps $f : X \to Y$, $g : Y \to Z$, and $h : Z \to W$ with a corresponding nullhomotopies $F : CX \to Z$ and $G : CY \to W$ called Toda system.

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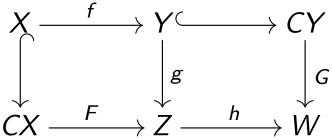

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$$
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X & \xrightarrow{f} & Y & \hookrightarrow & CY \\
\downarrow & & \downarrow g & & \downarrow G \\
CX & \xrightarrow{F} & Z & \xrightarrow{h} & W \\
\downarrow & & \downarrow & & \| \\
\Sigma X & \xrightarrow{\alpha} & \mathrm{hcof}(f, g, F) & \xrightarrow{\beta} & W
\end{array}
$$

and we apply the homotopy cofiber to get the Toda bracket.

Diagrammatic description for Toda system (cont.)

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$$X \xrightarrow{f} Y \xrightarrow{g} Z \qquad\qquad Y \xrightarrow{g} Z \xrightarrow{h} W$$

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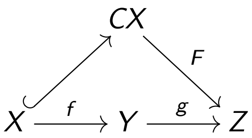

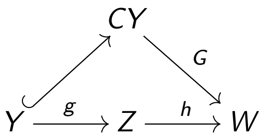

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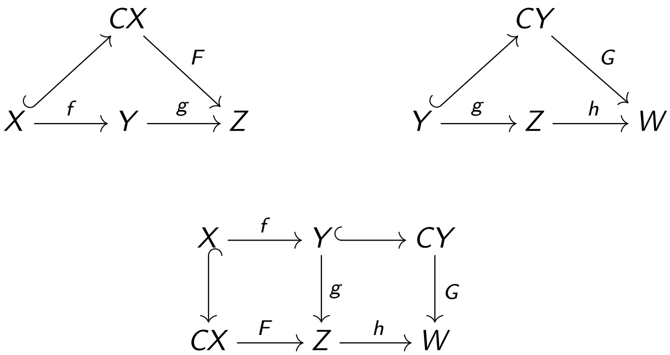

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$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \hookrightarrow & CY \\ \big\downarrow & & \big\downarrow g & & \big\downarrow G \\ CX & \xrightarrow{F} & Z & \xrightarrow{h} & W \end{array} \qquad \begin{array}{ccccc} X & \xrightarrow{g} & Z & \hookrightarrow & CZ \\ \big\downarrow & & \big\downarrow h & & \big\downarrow H \\ CY & \xrightarrow{G} & W & \xrightarrow{k} & V \end{array}$$

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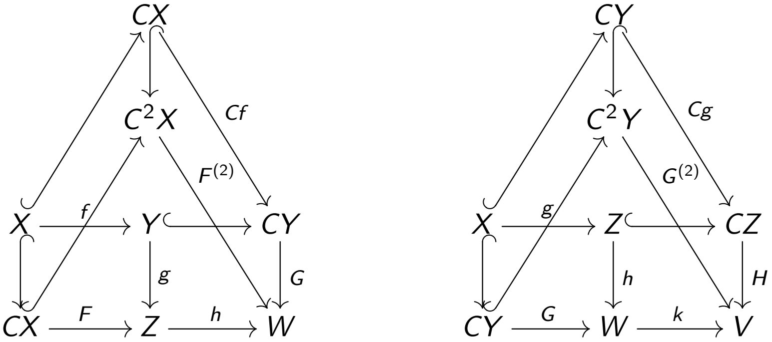

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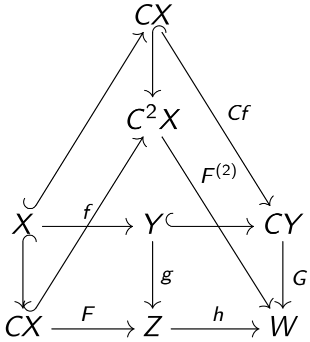

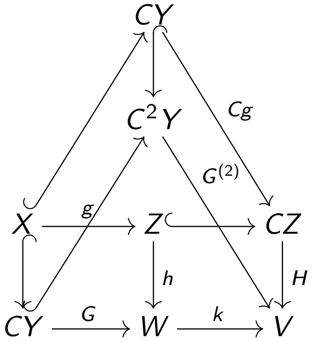

second order nullhomotopies $F^{(2)} : C^2X \to W, \quad G^{(2)} : C^2X \to V$

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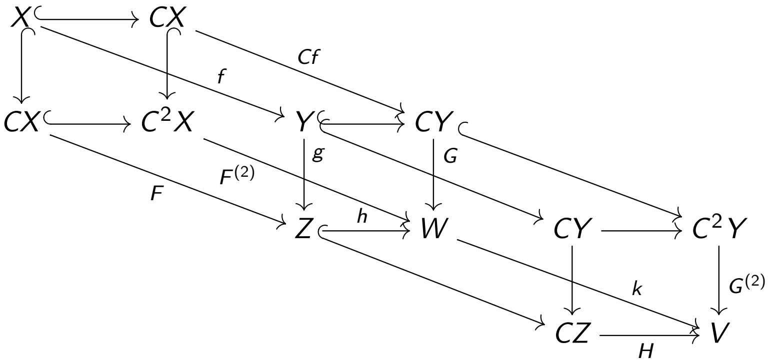

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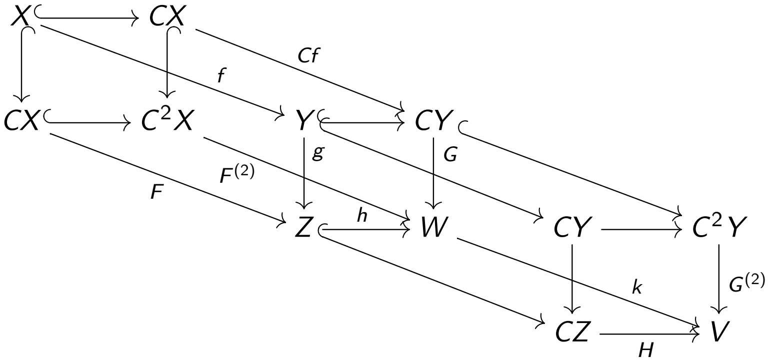
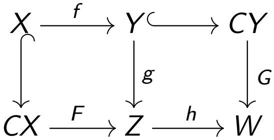
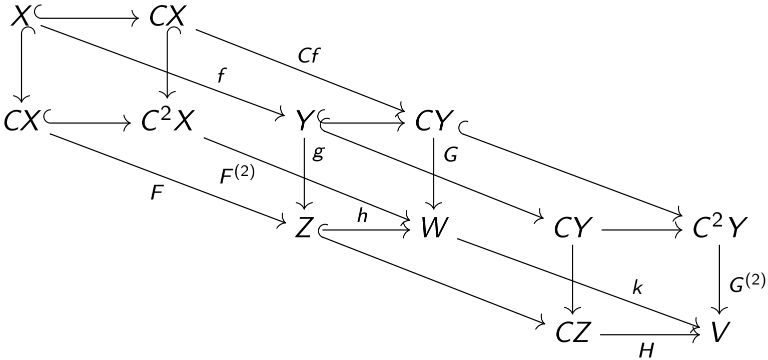


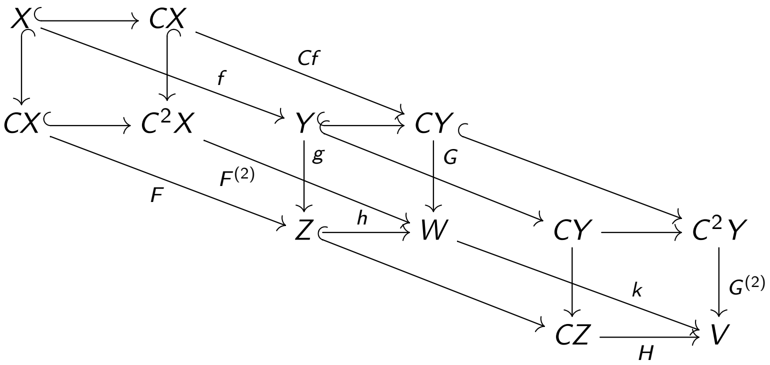


Diagrammatic description for Toda system(cont.)



This point of view allows us to define an equivalence relation between Toda systems (Two systems are equivalent if there is a zig-zag of weak equivalences between the corresponding diagrams).

Diagrammatic description for Toda system(cont.)



This point of view allows us to define an equivalence relation between Toda systems (Two systems are equivalent if there is a zig-zag of weak equivalences between the corresponding diagrams).

Two equivalent Toda systems induce equivalent Toda brackets.

Homotopy cofiber of a square

The *homotopy cofiber* of the square

$$\begin{array}{ccc} X & \xrightarrow{h} & Z \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[k]{} & W \end{array}$$

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$$\begin{array}{ccc} X & \xrightarrow{h} & Z \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ Y & \xrightarrow[k]{} & W \end{array}$$

is the colimit of